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Contract Nonr. 736 (00)   Project NR 330-027

TECHNICAL REPORT 3

June, 1954

TABLES OF SCATTERING FUNCTIONS FOR SPHERICAL COLLOIDAL PARTICLES II

(α=8.0(1.0)15 ; m = 1.15, 1.20, 1.25)

Computation Laboratory

Wayne University

Submitted by

Wilfried Heller   Project Director

Chemistry Department   Wayne University

Detroit, Michigan

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OFFICE OF NAVAL RESEARCH

Contract Nonr-736(00)
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Research on the size and shape of large molecules and colloidal particles

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Technical Report #3

<u>Tables of Scattering Functions</u>

<u>for Spherical Colloidal Particles II</u>

(Range considered: α = 8.0(1.0)15.0; m = 1.15, 1.20, 1.25)

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Wilfried Heller
Chemistry Department
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Detroit, Michigan

June 10, 1954

INTRODUCTORY REMARKS

The preceding Technical Report No. 2 contained light scattering functions computed by William J. Pangonis by means of a Marchant desk calculator. The procedure followed had been outlined in the Technical Report No. 1. The data obtained cover the entire colloidal range, and it is possible with them to determine the particle size in any monodisperse colloidal solution or emulsion of spheres. In case of heterodisperse systems the mean particle size is obtained. In the latter case, not only the mean particle diameter but also the distribution of diameters or, at least, the standard deviation are of great interest. The scattering functions previously compiled are not sufficient for this purpose. It is then necessary to have available functions extending into the microscopic range, for reasons explained in detail in the forthcoming Technical Report No. 4.

To carry out calculations of such functions for -values in excess of 7.0 would require a prohibitive amount of time and corresponding expense if it were done by desk calculator. Fortunately the Computation Laboratory of Wayne University, directed by Professor Arvid Jacobson, consented to carry out these computations at a compensation representing only a fraction of the actual cost. In addition to making it possible now to proceed with a systematic treatment of the problem of size distribution in heterodisperse systems, the new computations allow **now** the determination of **mean** diameters up to 1.96 microns in dispersions of spherical particles assuming the use of light of the wave length of 5,461 A.U. Since this includes the size of many bacteria and micro-organisms found in both ordinary and in sea water, the present extension of the light scattering functions suggests the application of the results to problems outside of the originally intended scope of this work.

The numbering of the pages in the present report follows the last numbers in the preceding report, so that the ulterior combination of the two reports into one volume will be possible without extensive changes. With this in mind, the last page of this report contains the summations $\sum_{n=1}^{\infty} \cdot \frac{|a_n|^2 + |b_n|^2}{2n+1}$ for both the data in this and in the preceding report.

Professor Arthur F. Stevenson again helped effectively on theoretical problems involved in the preparation of pertinent phases of this work.

Wilfried Heller

Data Obtained With the Electronic Computer

The table of values of the light scattering functions for spherical colloidal particles for α ranging from 8 to 15 at unit intervals and m taking the values 1.15, 1.20 and 1.25 was computed by the Computation Laboratory of Wayne University on the UDEC (Unitized Digital Electronic Computer).

The problem involved the solution of a set of equations based upon the Mie* theory of light scattering. The basic equations are:

$$(1)\quad J_{\perp} = \frac{\lambda^2}{4\pi^2 r^2}\left|\sum_{1}^{\infty} \frac{A_n P_n^{(1)}(\cos\gamma)}{\sin\gamma} + B_n \frac{d}{d\gamma} P_n^{(1)}(\cos\gamma)\right|^2$$

$$(2)\quad J_{\parallel} = \frac{\lambda^2}{4\pi^2 r^2}\left|\sum_{1}^{\infty} A_n \frac{d}{d\gamma} P_n^{(1)}(\cos\gamma) + \frac{B_n P_n^{(1)}(\cos\gamma)}{\sin\gamma}\right|^2$$

$$(3)\quad R = \frac{\lambda^2}{2\pi} \sum_{1}^{\infty} \frac{|a_n|^2 + |b_n|^2}{(2n+1)}$$

Equation (1) and (2) which concern $J_{\perp}$ the intensity of the component whose electric vector is perpendicular to the plane of observation and $J_{\parallel}$ the intensity of the component whose electric vector lies in the plane of observation were not evaluated. However, A_n and B_n, which refer to the electric and magnetic partial waves and are essential elements in the first two equations, were computed. The summation portion of equation (3) which will hereafter be designated as: $\sum = \sum \frac{|a_n|^2 + |b_n|^2}{(2n+1)}$ was the main portion of the computation.

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*Gustav Mie; "Beitraege zur Optik trueber Medien," Ann. Phys., 25, 377 (1908).

The notation used is the same as in Technical Report #2.

The breakdown of the equation is:

(4) $A = \dfrac{a_n}{n(n+1)}$ $\qquad B_n = \dfrac{b_n}{n(n+1)}$

(5) $$a_n = \frac{(-1)^{n+\frac{1}{2}}(2n+1)\left[m\,S_n'(\alpha)\,S_n(m\alpha)-S_n(\alpha)\,S_n'(m\alpha)\right]}{mS_n'(\alpha)S_n(m\alpha)-S_n(\alpha)S_n'(m\alpha)+(-1)^{n+\frac{1}{2}}\text{ (see below)}}$$

Con't----$\left[m\,S_{-n}'(\alpha)S_n(m\alpha)-S_{-n}(\alpha)\,S_n'(m\alpha)\right]$

(6) $$b_n = \frac{(-1)^{n+\frac{1}{2}}(2n+1)\left[S_n'(\alpha)S_n(m\alpha)-mS_n(\alpha)S_n'(m\alpha)\right]}{mS_n(\alpha)S_n'(m\alpha)-S_n'(\alpha)S_n(m\alpha)+(-1)^{n+\frac{1}{2}}\left[mS_{-n}(\alpha)S_n'\text{ (see below)}\right.}$$

Con't-----$\left.(m\alpha)-S_{-n}'(\alpha)S_n(m\alpha)\right]$

(7) $S_n(\alpha) = \sqrt{\frac{\pi\alpha}{2}}\; J_{n+\frac{1}{2}}(\alpha) = \alpha\, I_n(\alpha)$

$$S_{-n}(\alpha) = \sqrt{\frac{\pi\alpha}{2}}\,J_{-(n+\frac{1}{2})}(\alpha) = I_{-n}(\alpha)$$

$$S_n'(\alpha) = S_{n-1}(\alpha) \frac{-n}{\alpha} S_n(\alpha)$$

$$S_{-n}'(\alpha) = -S_{-(n-1)}(\alpha) \frac{-n}{\alpha} S_{-n}(\alpha)$$

The Mathematical Tables Project* has prepared tables of the sperical Bessel functions which are denoted by $I_n(x)$ and $I_{-n}(x)$.

The equations as they now stand were not in the proper form for the most economical operation of the UDEC digital computer.

Let $N_n = m\,S_n'(\alpha)\;S_n(m\alpha)--S_n(\alpha)\;S_n'(m\alpha)$

$D_n = (-1)^n\, m\, S_{-n}'(\alpha)\;S_n(m\alpha)--S_{-n}(\alpha)\;S_n'(m\alpha)$

Form the ratio and divide numerator and denominator of the right hand side by $S_n(m\alpha)\;S_{-n}(\alpha)$ and rearrange, then:

* Mathematical Tables Project, Tables of the Spherical Bessel Functions (New York: Columbia Press, 1945).

$$(8)\quad \frac{N_n}{D_n} = (-1)^{n+1}\left[\frac{m\left[\dfrac{S_{n-1}(\alpha)}{S_n(\alpha)} + \dfrac{S_{-(n-1)}(\alpha)}{S_{-n}(\alpha)}\right]}{m\,\dfrac{S_{-(n-1)}(\alpha)}{S_{-n}(\alpha)} + \dfrac{S_{n-1}(m\alpha)}{S_n(m\alpha)} + \dfrac{n(m^2-1)}{m\,\alpha}} - 1\right]\frac{S_n(\alpha)}{S_{-n}(\alpha)}$$

Let $N_n^* = S_n'(\alpha)\, S_n(m\alpha) - mS_n(\alpha)\, S_n'(m\alpha)$

$D_n^* = (-1)^n \quad mS_{n-1}(\alpha)S_n'(m\alpha) - S_{-n}'(\alpha)S_n(m\alpha)$

Form the ratic and divide numerator and denominator by $S_n(m\alpha)\, S_{-n}(\alpha)$ and rearrange, then:

$$(9)\quad \frac{N_n^*}{D_n^*} = (-1)^{n+1}\left[\frac{\dfrac{S_{n-1}(\alpha)}{S_n(\alpha)} + \dfrac{S_{-(n-1)}(\alpha)}{S_{-n}(\alpha)}}{\dfrac{mS_{n-1}(m\alpha)}{S_n(m\alpha)} + \dfrac{S_{-(n-1)}(\alpha)}{S_{-n}(\alpha)}} - 1\right]\frac{S_n(\alpha)}{S_{-n}(\alpha)}$$

Reference to equations (7) will give an indication of the labor saved by forming ratios. The argument cancels and the Bessel function table values may be used directly. The numerator element for $\frac{N_n}{D_n}$ and $\frac{N_n^*}{D_n^*}$ is the same and was stored on the drum and used twice. Similarly, all ratios were stored and used twice.

Upon substituting the $\frac{N_n}{D_n}$ and $\frac{N_n^*}{D_n^*}$ in equations (5) and (6) and dividing each equation by $n(n+1)$ we obtain the simplified expressions

$$(10)\quad A_n = (-1)^{n+1}\frac{(2n+1)}{n(n+1)}\left[\frac{\dfrac{-N_n}{D_n} - \left(\dfrac{N_n}{D_n}\right)^2 i}{1 + \left(\dfrac{N_n}{D_n}\right)^2}\right] = R\,(A_n) + I\,(A_n)$$

$$(11)\quad B_n = (-1)^{n+1} \frac{(2n+1)}{n(n+1)} \left[\frac{\frac{N_n^*}{D_n^*} + \left(\frac{N_n^*}{D_n^*}\right)^2 i}{1 + \left(\frac{N_n^*}{D_n^*}\right)^2} \right] = R\ (B_n) + I\ (B_n)$$

Where $R\ (A_n)$, $R\ (B_n)$, $I(A_n)$, and $I(B_n)$ are the real and imaginary parts of A_n and B_n.

The sperical Bessel terms required were formed during the computation by using the recurrence relations:

$$S_n(\alpha) = \frac{(2n-1)}{\alpha} S_{n-1}(\alpha) - S_{n-2}(\alpha)$$

$$S_{-n}(\alpha) = \frac{(sn-1)}{\alpha} S_{-(n-1)}(\alpha) - S_{-(n-2)}(\alpha)$$

The computation of $\frac{|a_n|^2}{(2n+1)}$ and $\frac{|b_n|^2}{(sn+1)}$ which are elements of $\sum$ was most economically performed by using the relations:

$$\frac{|a_n|^2}{(2n+1)} = n(n+1)\ I\ (A_n)$$

$$\frac{|b_n|^2}{(sn+1)} = n(n+1)\ I\ (B_n)$$

The number of significant figures for the Bessel functions used in the computation put an upper limit on the number of significant figures to be considered in the final results. Not more than 5 significant figures are correct--the sixth figure being in doubt.

During the machine computation the value of the ratio $\frac{S_{-(n-1)}(m\alpha)}{S_{-n}(m\alpha)}$ for $\alpha = 15$ and $m = 1.15$. exceeded the storage capacity of the computer.

This was due to the fact that S_{mn} (17.25) was near a root of the function and had an extremely small value which made the value of the ratio extremely large. This particular part of the problem was then calculated by hand. The values for $\alpha = 8$ and $m = 1.15$ were also hand calculated for comparison purposes using nine significant figures. The results indicated that only five significant machine figures were correct, as previously stated.

Interpolation for intermediate values of $\sum$ for any α within the range of the table may be accomplished by using the method of least squares for a polynomial of degree three if one wishes to utilize all five significant figures. The range for this accuracy must, however, be small.

Approximately two hours were required for the machine computation.

The following personnel should be given credit for the computation of this part of the tables:

Wesley Dixon
Lyle Langdon
James McCarty

m = 1.15

$\alpha = 8$

n	R(A_n)	I(A_n)	R(B_n)	I(B_n)	$\lvert a_n\rvert^2/2n+1$	$\lvert b_n\rvert^2/2n+1$
1	.509055	-1.300871	-.543383	1.267083	2.601743	2.534159
2	-.302348	.703380	.317831	-.686105	4.220282	4.116614
3	.250853	-.440563	-.212142	.491832	5.286764	5.901962
4	-.190567	.344679	.215401	-.290142	6.893598	5.802831
5	.175896	-.235167	-.172149	.246593	7.055032	7.397794
6	-.154731	.149982	.148891	-.197175	6.299280	8.281358
7	.111688	-.059983	-.109637	.056993	3.359085	3.191592
8	-.042488	.007905	.027902	-.003343	.569221	.240735
9	.009784	-.000454	-.004728	.000105	.040895	.009533
10	-.001760	.000016	.000671	-.000002	.001785	.000260
11	.000259	-.000000	-.000080		.000051	.000004
12	-.000032		.000008		.000001	
13	.000002					

$\alpha = 9$

n	R(A_n)	I(A_n)	R(B_n)	I(B_n)	$\lvert a_n\rvert^2/2n+1$	$\lvert b_n\rvert^2/2n+1$
1	.421026	-1.370720	-.255522	1.455132	2.741440	2.910262
2	-.181575	.791711	.246570	-.752620	4.750266	4.515708
3	.172793	-.526639	-.164536	.532547	6.319677	6.390563
4	-.164350	.378697	.133519	-.406108	7.573948	8.122157
5	.137286	-.304867	-.164628	.264176	9.146033	7.925251
6	-.139504	.221889	.134806	-.230856	9.319365	9.695964
7	.132962	-.150219	-.121295	.190815	8.412294	10.685646
8	-.104442	.063016	.104777	-.063607	4.537196	4.579695
9	.042137	-.008772	-.029458	.004192	.789491	.377341
10	-.010240	.000550	.005343	-.000149	.060549	.016449
11	.001975	-.000022	-.000827	.000003	.002957	.000518
12	-.000318	.000000	.000108		.000099	.000011
13	.000043		-.000011		.000002	.000000
14	-.000004		.000001			

$\alpha = 10$

n	R(A_n)	I(A_n)	R(B_n)	I(B_n)	$\lvert a_n\rvert^2/2n+1$	$\lvert b_n\rvert^2/2n+1$
1	.028237	-1.499468	-.218292	1.467548	2.998936	2.935093
2	-.143986	.807682	.030581	-.832209	4.846096	4.993256
3	.057481	-.577616	-.125446	.554983	6.931394	6.659795
4	-.098660	.427229	.077732	-.436147	8.544593	8.722945
5	.106257	-.332746	-.082418	.347100	9.982408	10.413019
6	-.098662	.274019	.126659	-.243758	11.508814	10.237836
7	.110165	-.210098	-.105356	.216649	11.765495	12.132366
8	-.113673	.149952	.097725	-.184321	10.796592	13.271132
9	.098136	-.066559	-.099594	.070571	5.990351	6.351457
10	-.042063	.009759	.031054	-.005188	1.073505	.570671
11	.010678	-.000656	-.005949	.000203	.086646	.026842
12	-.002182	.000029	.000987	-.000006	.004637	.000948
13	.000378	-.000000	-.000142	.000000	.000175	.000024
14	-.000055		.000017		.000004	.000000
15	.000006		-.000001			

M = 1.15

n	R(A_n)	I(A_n)	R(B_n)	I(B_n)	$\|a_n\|^2/2n+1$	$\|b_n\|^2/2n+1$
				α = 11		
1	-.069652	-1.496761	.136444	1.487486	2.993522	2.974967
2	.071034	.827237	-.002034	-.833328	4.963423	4.999969
3	.028978	-.581890	.044890	.579861	6.982690	6.958332
4	.005261	.449938	.055354	-.443090	8.998769	8.861790
5	.049313	-.359912	-.022733	.365252	10.797373	10.957587
6	-.064564	.295426	.046226	-.302459	12.407923	12.703294
7	.069189	-.248616	-.096864	.226459	13.922506	12.681706
8	-.085862	.199084	.081510	-.203471	14.334119	14.649963
9	.096217	-.149055	-.077290	.177451	13.414959	15.970655
10	-.092125	.070375	.093817	-.077777	7.741250	8.555471
11	.042151	-.010870	-.032670	.006352	1.434848	.838529
12	-.011117	.000774	.006562	-.000268	.120794	.041960
13	.002385	-.000038	-.001154	.000008	.006981	.001633
14	-.000440	.000001	.000179	-.000000	.000295	.000049
15	.000069	.000000	-.000024	.000000	.000009	.000001
16	-.000009		.000002		.000000	.000000
17	.000001		.000000			
18	.000000					
				α = 12		
1	-.266125	-1.451206	.362206	1.406745	2.902413	2.813475
2	.170516	.796876	-.144992	-.807297	4.781257	4.843773
3	-.101052	-.565275	.086031	.570363	6.783304	6.844346
4	.033465	.447498	-.077512	-.436233	8.949977	8.724662
5	-.040779	-.362073	-.009229	.366434	10.862221	10.993030
6	-.014072	.308883	-.013812	-.308906	12.973088	12.974074
7	.033496	-.263601	-.019238	.266468	14.761671	14.922245
8	-.045954	.226803	.072731	-.211047	16.329848	15.195392
9	.065368	-.188443	-.061875	.191092	16.959924	17.198334
10	-.080163	.147332	.059473	-.170138	16.206574	18.715186
11	.086208	-.074429	-.087128	.084957	9.824628	11.214320
12	-.042421	.012138	.034331	-.007722	1.893659	1.204714
13	.011551	-.000904	-.007176	.000347	.164672	.063301
14	-.002583	.000048	.001325	-.000012	.010151	.002672
15	.000503	-.000001	-.000218	.000000	.000471	.000088
16	-.000084	.000000	.000031		.000016	.000002
17	.000012		-.000004		.000000	.000000
18	-.000002		.000001			
19	.000000					
				α = 13		
1	-.574253	-1.232669	.422425	1.369849	2.465339	2.739683
2	.225875	.766827	-.305564	-.700048	4.600963	4.200288
3	-.191447	-.511767	.148249	.542881	6.141205	6.514563
4	.114071	.418941	-.123070	-.413360	8.378836	8.267190
5	-.069236	-.353092	.092649	.341536	10.592780	10.246074
6	.061889	.296612	-.022557	-.307871	12.457744	12.930595
7	-.011825	-.267334	.038442	.262223	14.970717	14.684517
8	-.009848	.235699	-.001495	-.236101	16.970378	16.999318
9	.027263	-.207529	-.052812	.196959	18.677681	17.726324
10	-.047932	.178014	.045527	-.179364	19.581597	19.730067
11	.065343	-.144749	-.043928	.162365	19.106919	21.432264
12	-.080149	.078607	.079289	-.091720	12.262739	14.308324
13	.042785	-.013573	-.036036	.009340	2.470440	1.699941
14	-.012010	.001051	.007802	-.000442	.220891	.092852

$m_2 = 1.15$

$\alpha = 13$ (cont.)

n	$R(A_n)$	$I(A_n)$	$R(B_n)$	$I(B_n)$	$\lvert a_n\rvert^2/2n+1$	$\lvert b_n\rvert^2/2n+1$
15	.002780	-.000059	-.001503	.000017	.014369	.004199
16	-.000567	.000002	.000262	-.000000	.000722	.000153
17	.000101	.000000	-.000040		.000027	.000004
18	-.000016		.000005		.000000	.000000
19	.000002		-.000001			
20	.000000					

$\alpha = 14$

n	$R(A_n)$	$I(A_n)$	$R(B_n)$	$I(B_n)$	$\lvert a_n\rvert^2/2n+1$	$\lvert b_n\rvert^2/2n+1$
1	-.603605	-1.195518	.686317	1.052597	2.391037	2.105363
2	.376836	.594550	-.319836	-.683744	3.567302	4.102445
3	-.211974	-.492055	.258991	.425916	5.904670	5.110976
4	.186398	.351071	-.149430	-.393247	7.021432	7.864934
5	-.119966	-.322014	.136570	.305666	9.660437	9.169971
6	.089000	.281399	-.099318	-.273468	11.818760	11.485672
7	-.074433	-.245271	.044560	.260233	13.735214	14.573066
8	.030947	.231984	-.054849	-.222601	16.702878	16.027325
9	-.008346	-.210780	.017650	.209625	18.970262	18.866329
10	-.012040	.190147	.036141	-.183806	20.916198	20.218709
11	.033034	-.167737	-.031824	.168225	22.141365	22.205736
12	-.051698	.141347	.030432	-.154252	22.050234	24.063402
13	.073717	-.082712	-.070338	.097783	15.053602	17.796499
14	-.043249	.015211	.037823	-.011271	3.194388	2.367018
15	.012478	-.001216	-.008449	.000554	.291945	.133191
16	-.002977	.000073	.001687	-.000023	.019872	.006384
17	.000632	-.000003	-.000307	.000000	.001070	.000253
18	-.000119	.000000	.000050		.000044	.000008
19	.000020		-.000008		.000001	.000000
20	-.000002		.000000		.000000	
21	.000000					

$\alpha = 15$

n	$R(A_n)$	$I(A_n)$	$R(B_n)$	$I(B_n)$	$\lvert a_n\rvert^2/2n+1$	$\lvert b_n\rvert^2/2n+1$
1	-.731936	-.914672	.729215	.926672	1.829344	1.853326
2	.397874	.540639	-.406654	-.508208	3.243836	3.049224
3	-.283764	-.359314	.268698	.405221	4.311773	4.862645
4	.197242	.333282	-.218006	-.280724	6.665649	5.614483
5	-.171404	-.248391	.147249	.292592	7.451739	8.777761
6	.121187	.251049	-.136857	-.227043	10.544087	9.535805
7	-.098469	-.224759	.101278	.221568	12.586540	12.407843
8	.081409	.203585	-.059399	-.220085	14.658135	15.846135
9	-.044869	-.201107	.065318	.188476	18.099709	16.962909
10	.022370	.188251	-.030253	-.185992	20.707698	20.459193
11	-.000431	-.174241	-.022078	.171399	22.999856	22.624788
12	-.02031	.1576	.02029	-.1576	24.59	24.59
13	-.058682	-.119547	.058074	.120353	21.757590	21.904363
14	-.066800	.086566	.060324	-.102694	18.178902	21.565764
15	.043750	-.017071	-.039591	.013555	4.097160	3.253284
16	-.012971	.001402	.009122	-.000689	.381358	.187518
17	.003174	-.000088	-.001875	.000030	.026952	.009413
18	-.000698	.000004	.000355	-.000001	.001544	.000400
19	.000137	-.000000	-.000061	.000000	.000070	.000014
20	-.000024		.000009		.000002	.000000
21	.000003		-.000001		.000000	
22	-.000000		.000000			

m = 1.20

$\alpha = 8$

n	$R(A_n)$	$I(A_n)$	$R(B_n)$	$I(B_n)$	$\|a_n\|^2/2n+1$	$\|b_n\|^2/2n+1$
1	.053813	-1.498067	.092281	1.494301	2.996135	2.988601
2	.031875	.832112	.091314	-.823208	4.992677	4.939249
3	.101261	-.565202	.003621	.583310	6.782426	6.999729
4	-.073431	.437691	.119674	-.415559	8.753833	8.311176
5	.101310	-.336160	-.119959	.322013	10.084802	9.660391
6	-.129697	.239232	.088881	-.281467	10.047764	11.821594
7	.133353	-.121494	-.133374	.146370	6.803676	8.196741
8	-.061852	.017499	.045921	-.009288	1.259947	.668755
9	.013814	-.000907	-.007210	.000246	.081686	.022187
10	-.002401	.000030	.000983	-.000005	.003323	.000556
11	.000348	-.000000	-.000114		.000092	.000009
12	-.000041		.000011		.000001	
13	.000004					

$\alpha = 9$

n	$R(A_n)$	$I(A_n)$	$R(B_n)$	$I(B_n)$	$\|a_n\|^2/2n+1$	$\|b_n\|^2/2n+1$
1	-.279403	-1.446051	.330682	1.423164	2.892103	2.846314
2	.129927	.812568	-.154408	-.803668	4.875413	4.821999
3	-.097827	-.566451	.030807	.581702	6.797423	6.980427
4	-.016420	.449400	-.065721	-.440194	8.988001	8.803896
5	.018629	-.365717	-.060764	.356314	10.971538	10.689415
6	-.056345	.298911	.076171	-.289497	12.554270	12.158890
7	.093127	-.230209	-.053119	.256830	12.891740	14.385320
8	-.117558	.129062	.110438	-.159875	9.292535	11.511039
9	.062315	-.020344	-.049868	.012521	1.830986	1.126973
10	-.014668	.001132	.008326	-.000363	.124598	.039987
11	.002719	-.000042	-.001224	.000008	.005604	.001136
12	-.000429	.000001	.000158	-.000000	.000179	.000024
13	.000057		-.000017		.000004	.000000
14	-.000006		.000001			

$\alpha = 10$

n	$R(A_n)$	$I(A_n)$	$R(B_n)$	$I(B_n)$	$\|a_n\|^2/2n+1$	$\|b_n\|^2/2n+1$
1	-.616244	-1.177502	.462681	1.340386	2.355004	2.680773
2	.251680	.748793	-.316663	-.687571	4.492760	4.125431
3	-.180714	-.520655	.170140	.528605	6.247866	6.343241
4	.126304	.411226	-.090140	-.431164	8.224521	8.623270
5	-.035594	-.363179	.098149	.338210	10.895388	10.146295
6	.017933	.308481	.018319	-.308435	12.956239	12.954305
7	.022494	-.265956	-.043261	.260680	14.893558	14.598113
8	-.061650	.218747	.025399	-.233347	15.749804	16.801001
9	.101032	-.136207	-.084509	.168809	12.258665	15.192836
10	-.063036	.023769	.054051	-.016772	2.614689	1.845013
11	.015529	-.001395	-.009491	.000518	.184164	.068377
12	-.003033	.000057	.001482	-.000013	.008958	.002138
13	.000513	-.000001	-.000207	.000000	.000323	.000052
14	-.000074		.000024		.000008	.000000
15	.000009		-.000002		.000000	
16	-.000001					

m = 1.20

α = 11

n	R(A_n)	I(A_n)	R(B_n)	I(B_n)	$\lvert a_n\rvert^2/2n+1$	$\lvert b_n\rvert^2/2n+1$
1	-.651918	-1.121142	.740831	.868618	2,242285	1.737201
2	.405189	.513812	-.345441	-.649878	3.082873	3.898068
3	-.233971	- .465937	.274747	.389896	5.591249	4.678729
4	.187797	.348995	-.170900	-.371384	6.979904	7.427695
5	-.136970	- .305258	.118075	.323615	9.157760	9.708457
6	.067681	.293946	-.113436	-.260079	12.345765	10.923352
7	-.042744	- .260855	.013116	.267213	14.607894	14.963969
8	.003143	.236069	.018096	-.234716	16.996986	16.899606
9	.034855	- .205195	-.004063	.211032	18.467586	18.992962
10	-.083335	.142019	.057300	-.171797	15.622135	18.897715
11	.063979	- .027955	-.058312	.022383	3.690152	2.954651
12	-.016460	.001707	.010724	-.000720	.266385	.112409
13	.003351	- .000075	-.001757	.000020	.013778	.003790
14	-.000602	.000002	.000263	-.000000	.000551	.000105
15	.000094	.000000	-.000034		.000016	.000002
16	-.000013		.000003		.000000	.000000
17	.000001					
18	.000000					

α = 12

n	R(A_n)	I(A_n)	R(B_n)	I(B_n)	$\lvert a_n\rvert^2/2n+1$	$\lvert b_n\rvert^2/2n+1$
1	-.738341	-.616743	.749455	.784485	1.233487	1.568960
2	.413066	.471535	-.411243	-.348660	2.829214	2.091963
3	-.290723	-.267035	.281306	.363851	3.204423	4.426189
4	.211306	.302531	-.224988	-.227564	6.050639	4.551275
5	-.175736	-.235668	.163499	.266324	7.070041	7.989725
6	.137050	.226771	-.127730	-.242152	9.524403	10.170385
7	-.086351	-.236316	.117622	.197977	13.233745	11.086724
8	.059351	.220116	-.036452	-.230346	15.848380	16.584931
9	-.022431	-.208700	.001311	.211102	18.783075	18.999265
10	-.012492	.190088	-.012162	-.190131	20.909724	20.914483
11	.064558	-.145648	-.031081	.168514	19.225573	22.243900
12	-.064[illegible]19	.033[illegible]7	.062376	-.029813	5.149217	4.650943
13	.017436	-.002078	-.012062	.000986	.378209	.179585
14	-.003671	.000097	.002055	-.000030	.020503	.006411
15	.000692	-.000[illegible]3	-.000325	.000000	.000890	.000196
16	-.000115	.000000	.000046		.000029	.000004
17	.000017		-.000005		.000000	.000000
18	-.000002		.000001			
19	.000000		.000000			

α = 13

n	R(A_n)	I(A_n)	R(B_n)	I(B_n)	$\lvert a_n\rvert^2/2n+1$	$\lvert b_n\rvert^2/2n+1$
1	-.666750	-.406324	.677693	.428037	.812648	.856061
2	.376644	.238155	-.386904	-.261547	1.428934	1.569285
3	-.282942	-.220680	.263690	.166998	2.648167	2.003944
4	.209663	.143109	-.224702	-.212062	2.862195	4.241232
5	-.182996	-.195081	.175464	.129980	5.852459	3.899383
6	.154772	.157027	-.150628	-.190441	6.595160	7.998537
7	-.130017	-.166296	.126148	.179063	9.312624	10.027538
8	.095438	.187562	-.114251	-.147943	13.504497	10.651921
9	-.069900	-.184672	.053384	.196622	16.620514	17.695969
10	.036689	.183582	-.016312	-.189505	20.194080	20.845664
11	-.005700	-.174055	.024305	.170785	22.975365	22.543634
12	-.045204	.146294	.007929	-.159863	22.821937	24.938700
13	.065404	-.039158	-.065644	.039591	7.126829	7.205668

α = 13 (cont.)

n	$R(A_n)$	$I(A_n)$	$R(B_n)$	$I(B_n)$	$\|a_n\|^2/2n+1$	$\|b_n\|^2/2n+1$
14	-.018511	.002526	.013526	-.001336	.530470	.280720
15	.004001	-.000123	-.002368	.000043	.029752	.010426
16	-.000786	.000005	.000393	-.000001	.001385	.000346
17	.000138	-.000000	-.000059	.000000	.000051	.000009
18	-.000021		.000007		.000001	.000000
19	.000003		-.000001		.000000	
20	.000000		.000000			

α = 14

1	-.563397	-.254779	.410363	.122144	.509558	.244275
2	.256231	.088032	-.318333	-.147748	.528492	.886469
3	-.223305	-.104033	.206209	.085360	1.248405	1.024305
4	.185043	.096940	-.172819	-.080905	1.938816	1.618093
5	-.146857	-.073582	.170617	.116071	2.207483	3.482149
6	.151094	.121256	-.129168	-.069428	5.092760	2.915979
7	-.129916	-.101386	.133960	.133591	5.677664	7.481100
8	.118069	.118969	-.117519	-.129795	8.565790	9.345254
9	-.097468	-.146161	.105562	.107523	13.154576	9.677065
10	.075781	.153504	-.064969	-.165405	16.885501	18.194591
11	-.046910	-.160548	.027865	.169666	21.192420	22.395962
12	.020066	.157703	-.033212	-.153054	24.601713	23.876479
13	.026173	-.143584	.010892	.147547	26.132445	26.853706
14	-.065287	.046546	.066976	-.052136	9.774660	10.948654
15	.019677	-.003069	-.015139	.001799	.736734	.431833
16	-.004339	.000155	.002707	-.000060	.042251	.016438
17	.000834	-.000006	-.000467	.000001	.002091	.000585
18	-.000163	.000000	.000074	.000000	.000084	.000017
19	.000027		-.000011		.000002	.000000
20	-.000003		.000001		.000000	
21	.000000		-.000000			

α = 15

1	-.093798	-.005883	.351771	.087527	.011766	.175041
2	.209724	.056581	-.071276	-.006139	.339489	.036829
3	-.079708	-.011098	.162088	.049182	.133183	.590188
4	.128540	.040315	-.089451	-.018528	.806311	.370548
5	-.107578	-.034845	.108410	.035482	1.045375	1.064478
6	.096256	.033555	-.120037	-.057026	1.409345	2.395089
7	-.117993	-.070532	.087490	.032492	3.949841	1.819566
8	.103965	.062121	-.114855	-.090566	4.472743	6.520817
9	-.102962	-.082104	.104637	.091487	7.389415	8.233868
10	.094174	.111076	-.093331	-.075285	12.218371	8.281411
11	-.077972	-.125997	.071883	.136386	16.631617	18.003004
12	.053862	.139466	-.036668	-.151378	21.756808	23.615079
13	-.031021	-.141555	.039571	.136926	25.763134	24.920642
14	-.008486	.137572	-.025076	-.133381	28.890168	28.010152
15	.063925	-.055269	-.064535	.067291	13.264728	16.150069
16	-.020981	.003741	.016967	-.002420	1.017555	.658343
17	.004695	-.000192	-.003074	.000082	.059034	.025288
18	-.000984	.000008	.000549	-.000002	.003063	.000954
19	.000188	-.000000	-.000091	.000000	.000132	.000030
20	-.000033		.000013		.000004	.000000
21	.000005		-.000001		.000000	
22	-.000000		.000000			

$m = 1.25$

n	$R(A_n)$	$I(A_n)$	$R(B_n)$	$I(B_n)$	$\|a_n\|^2/2n+1$	$\|b_n\|^2/2n+1$
			$\alpha = 8$			
1	-.424466	-1.368343	.645047	1.132865	2.736686	2.265708
2	.326644	.675516	-.187513	-.788769	4.053099	4.732616
3	-.116342	-.559140	.192371	.510945	6.709692	6.131316
4	.065555	.440240	-.071541	-.438333	8.804804	8.766673
5	-.022216	-.365316	-.041669	.361873	10.959482	10.856194
6	-.053566	.299965	.009031	-.309260	12.598553	12.988929
7	.117434	-.198323	-.071129	.247420	13.106095	13.855564
8	-.084244	.035318	.073074	-.025329	2.542942	1.823738
9	.018440	-.001622	-.010542	.000527	.145980	.047478
10	-.003081	.000049	.001361	-.000009	.005470	.001067
11	.000439	-.000001	-.000155	.000000	.000146	.000018
12	-.000052		.000014		.000002	
13	.000004		-.000001			
			$\alpha = 9$			
1	-.745856	-.829630	.694274	1.034086	1.659261	2.068150
2	.361531	.624089	-.413116	-.471651	3.744539	2.829884
3	-.278953	-.377114	.216334	.487395	4.525378	5.848710
4	.155054	.388088	-.198906	-.330314	7.761767	6.606266
5	-.102947	-.335057	.110725	.329486	10.051716	9.884584
6	.057558	.298432	-.001969	-.309511	12.534158	12.999471
7	.009004	-.267554	.021270	.266158	14.983029	14.904892
8	-.082297	.202719	.022408	-.233966	14.595779	16.845609
9	.085440	-.043535	-.081168	.038073	3.918180	3.426633
10	-.019929	.002103	.012528	-.000825	.231367	.090802
11	.003524	-.000071	-.001721	.000017	.009412	.002244
12	-.000543	.000001	.000216	-.000000	.000287	.000045
13	.000072		-.000023		.000006	.000000
14	-.000008		.000001		.000000	

m = 1.25

$\alpha = 10$

n	$R(A_n)$	$I(A_n)$	$R(B_n)$	$I(B_n)$	$\lvert a_n \rvert^2/2n+1$	$\lvert b_n \rvert^2/2n+1$
1	-.728243	-.570644	.726035	.561566	1.141288	1.123128
2	.404171	.314817	-.414921	-.376823	1.888905	2.260939
3	-.290332	-.320939	.283480	.223036	3.851271	2.676412
4	.224775	.213772	-.208635	-.309248	4.275454	6.184932
5	-.163476	-.266323	.181397	.210263	7.989696	6.307883
6	.119649	.252949	-.128194	-.241522	10.623860	10.143954
7	-.078804	-.242232	.032418	.263877	13.565025	14.777152
8	.023812	.233686	-.041415	-.228612	16.825456	16.460057
9	.046763	-.200187	.015230	.210007	18.016887	18.900671
10	-.086117	.054190	.087369	-.056977	5.961000	6.267550
11	.021588	-.002715	-.014775	.001261	.358382	.166531
12	-.003979	.000098	.002123	-.000028	.015413	.004388
13	.000655	-.000002	-.000286	.000000	.000527	.000100
14	-.000093	.000000	.000034		.000013	.000001
15	.000011		-.000003		.000000	.000000
16	-.000001		.000000			
17	.000000					

$\alpha = 11$

n	$R(A_n)$	$I(A_n)$	$R(B_n)$	$I(B_n)$	$\lvert a_n \rvert^2/2n+1$	$\lvert b_n \rvert^2/2n+1$
1	-.615000	-.320482	.482226	.175491	.640965	.350964
2	.307767	.135716	-.346112	-.184578	.814298	1.107448
3	-.242519	-.129461	.249401	.140325	1.553540	1.683888
4	.215489	.160048	-.187171	-.100039	3.200965	2.000785
5	-.170419	-.115643	.183179	.192544	3.469300	5.776314
6	.153344	.176022	-.152526	-.128226	7.392951	5.385476
7	-.122953	-.187094	.129316	.168876	10.477283	9.457081
8	.090050	.194423	-.053512	-.223292	13.998489	16.077031
9	-.046567	-.200293	.054378	.196041	18.026375	17.643757
10	-.013574	.189939	-.040633	-.181832	20.893389	20.001617
11	.084977	-.067758	-.087042	.083397	8.944174	11.008454
12	-.023428	.003500	.017378	-.001907	.546070	.297501
13	.004454	-.000133	-.002569	.000044	.024348	.008098
14	-.000773	.000004	.000367	-.000000	.000909	.000205
15	.000118	-.000000	-.000047		.000026	.000004
16	-.000015		.000004		.000000	.000000
17	.000002		.000000			
18	.000000					

m = 1.25

n	R(A_n)	I(A_n)	R(B_n)	I(B_n)	$\|a_n\|^2/2n+1$	$\|b_n\|^2/2n+1$
			$\alpha = 12$			
1	-.084750	-.004802	.384451	.106024	.009605	.212037
2	.227376	.067437	-.082673	-.008283	.404625	,049703
3	-.108158	-.020776	.174005	.057562	.249314	.690733
4	.134306	.044453	-.128272	-.040110	.889079	.802216
5	-.144540	-.070538	.109556	.036302	2.116154	1.089065
6	.118959	.055732	-.149282	-.113686	2.340770	4.774807
7	-.131632	-.109064	.119083	.072635	6.107607	4.067615
8	.117034	.133830	-.117793	-.109775	9.635823	7.903843
9	-.094020	-.153589	.067588	.186658	13.823088	16.799280
10	.061075	.168819	-.062252	-.167838	18.570183	18.462213
11	-.014418	-.173041	.055701	.154117	22.841521	20.343473
12	-.080032	.084293	.072453	-.114418	13.149755	17.849277
13	.025550	-.004537	-.020489	.002883	.825741	.524826
14	-.004952	.000177	.003069	-.000068	.037338	.014329
15	.000897	-.000006	-.000459	.000001	.001497	.000392
16	-.000146	.000000	.000064	.000000	.000048	.000009
17	.000021		-.000008		.000001	.000000
18	-.000003		.000001		.000000	
19	.000000		.000000			
			$\alpha = 13$			
1	-.008625	-.000049	-.294726	.060283	.000099	.1205[illegible]2
2	-.138937	.023836	-.041175	-.002038	.143016	.0122[illegible]9
3	-.051631	-.004601	-.072448	.009132	.055222	.1095[illegible]1
4	-.007798	.000135	-.065300	-.009674	.002702	.1934[illegible]3
5	-.053927	-.008105	.038810	.004153	.243174	.1245[illegible]0
6	.081920	.023459	-.046286	-.007080	.985306	.2973[illegible]1
7	-.072353	-.021224	.112450	.061134	1.188594	3.4235[illegible]6
8	.103396	.061036	-.085415	-.036523	4.394599	2.6297[illegible]3
9	-.104587	-.091036	.097015	.063887	8.193312	5.7498[illegible]4
10	.092592	.118736	-.076059	-.153148	13.061026	16.8462[illegible]5
11	-.069125	-.140163	.066517	.143395	18.501542	18.9282[illegible]7
12	.035460	.151986	-.063303	-.129255	23.709855	20.1637[illegible]2
13	.068564	-.102512	-.038736	.137436	18.657220	25.0134[illegible]7
14	-.028023	.005937	.024268	-.004401	1.246796	.9242[illegible]8
15	.005491	-.000233	-.003634	.000102	.056095	.0245[illegible]6
16	-.001028	.000008	.000565	-.000002	.002373	.0007[illegible]6
17	.000177	-.000000	-.000082	.000000	.000084	.000[illegible]18
18	-.000027		.000010		.000002	.0000[illegible]0
19	.000004		-.000001		.000000	
20	-.000000		.000000			

m = 1.25

$\alpha = 14$

n	$R(A_n)$	$I(A_n)$	$R(B_n)$	$I(B_n)$	$\|a_n\|^2/2n+1$	$\|b_n\|^2/2n+1$
1	.546684	−.236501	−.411308	.122820	.473002	.245633
2	−.195401	.048658	.300557	−.128022	.291953	.768116
3	.197243	−.076765	−.098266	.017050	.921183	.204590
4	−.049491	.005508	.139041	−.048073	.110167	.961456
5	.074887	−.015983	−.009133	.000227	.479498	.006825
6	−.006317	.000128	.024349	−.001927	.005415	.080933
7	−.029660	−.003323	−.003972	.000058	.186128	.003298
8	.031980	.004413	−.076193	−.027857	.317803	2.005746
9	−.072469	−.028782	.053828	.014754	2.590437	1.327893
10	.087548	.057412	−.070503	−.031088	6.315381	3.419726
11	−.087140	−.089165	.079748	.122238	11.769780	16.135546
12	.072299	.114685	−.068228	−.122174	17.890985	19.059289
13	−.049345	−.129567	.066014	.108014	23.581247	19.658624
14	−.048241	.118455	−.003394	−.138011	24.875592	28.982479
15	.030925	−.007878	−.028957	.006855	1.890812	1.645275
16	−.006074	.000304	.004278	−.000151	.082896	.041079
17	.001168	−.000011	−.000683	.000004	.003655	.001251
18	−.000209	.000000	.000104	−.000000	.000139	.000034
19	.000034		−.000015		.000004	.000000
20	−.000004		.000001		.000000	
21	.000000		−.000000			

$\alpha = 15$

n	$R(A_n)$	$I(A_n)$	$R(B_n)$	$I(B_n)$	$\|a_n\|^2/2n+1$	$\|b_n\|^2/2n+1$
1	.701412	−.484107	−.689885	.455317	.968215	.910602
2	−.383042	.252655	.373701	−.232270	1.515930	1.393621
3	.240800	−.127036	−.268068	.176654	1.524441	2.119843
4	−.201984	.125747	.157858	−.064624	2.514944	1.292476
5	.105649	−.033485	−.159542	.092952	1.004573	2.788569
6	−.103419	.045295	.058857	−.011628	1.902406	.488391
7	.049341	−.009416	−.065471	.017091	.527315	.957110
8	−.011668	.000577	.041725	−.007616	.041604	.548409
9	.001471	−.000010	.042549	.008953	.000923	.805783
10	.041746	.009612	−.025932	−.003589	1.057372	.394856
11	−.067634	−.032186	.041942	.010752	4.248645	1.419295
12	.078567	.064249	−.079024	−.093601	10.022938	14.601847
13	−.071909	−.092439	.068087	.103624	16.823916	18.859599
14	.057041	.107982	−.065710	−.090268	22.676283	18.956284
15	.020207	−.125927	.035200	.118731	30.222497	28.495483
16	−.034349	.010658	.034920	−.011056	2.899124	3.007248
17	.006722	−.000396	−.005029	.000221	.121264	.067753
18	−.001318	.000016	.000816	−.000006	.005498	.002108
19	.000245	−.000000	−.000130	.000000	.000222	.000062
20	−.000041		.000019		.000007	.000001
21	.000006		−.000002		.000000	.000000
22	−.000000		.000000			

$$\sum_{n=1}^{\infty} \cdot \frac{|a_n|^2 + |b_n|^2}{2n+1}$$

$\alpha \backslash m$	1.05	1.10	1.15	1.20	1.25	1.30
.20	$.0_{7}9151$	$.0_{6}3622$	$.0_{6}7961$	$.0_{5}1384$	$.0_{5}2114$	$.0_{8}2971$
.40	$.0_{5}5636$	$.0_{4}2228$	$.0_{4}4943$	$.0_{4}8621$	$.0_{3}1327$	$.0_{3}1884$
.60	$.0_{4}5989$	$.0_{3}2387$	$.0_{3}5340$	$.0_{3}9419$	$.0_{2}1458$	$.0_{2}2077$
.80	$.0_{3}3044$	.001228	.002773	.004938	.007707	.011071
1.00	.001028	.004171	.009492	.017236	.026806	.038744
1.20	.002645	.010786	.024653	.044406	.070082	.101656
1.40	.005637	.023010	.052622	.094726	.149358	.216399
1.60	.010572	.042878	.097324	.175430	.276368	.400754
1.80	.017754	.072302	.165137	.297592	.471745	.691326
2.00	.028167	.115122	.264523	.481872	.774470	1.15292
2.20	.042825	.176504	.410219	.755720	1.22566	1.82623
2.40	.063336	.262713	.615471	1.13663	1.83020	2.67696
2.60	.090590	.379073	.887194	1.62271	2.56650	3.67580
2.80	.126408	.526707	1.22611	2.21230	3.45402	4.92732
3.00	.171496	.712506	1.63848	2.93395	4.57619	6.65140
3.20	.226729	.936254	2.14272	3.83175	5.99483	8.54654
3.40	.293356	1.20748	2.76241	4.94494	7.67492	10.7429
3.60	.373227	1.53650	3.51211	6.24996	9.56501	13.1698
3.80	.468645	1.92905	4.39521	7.74486	11.7190	16.0558
4.00	.581919	2.39236	5.42118	9.46429	14.2457	19.4430
4.20	.715009	2.93032	6.59267	11.4532	17.2666	23.0333
4.40	.869527	3.54894	7.94182	13.7122	20.2683	26.7432
4.60	1.04724	4.25482	9.47387	16.2177	23.6699	30.9505
4.80	1.24898	5.05748	11.1843	18.9983	27.5105	35.7844
5.00	1.47808	5.96307	13.1253	22.1247	31.8126	40.6523
5.20	1.73581	6.98761	15.2874	25.6027	36.3190	45.2973
5.40	2.02632	8.12857	17.7066	29.4075	40.9053	50.3798
5.60	2.37049	9.40764	20.3487	33.3022	45.8653	56.2933
5.80	2.71832	10.8227	23.2443	37.5639	51.4870	62.3738
6.00	3.12212	12.3741	26.3219	42.3511	57.4078	67.6542
6.20	3.66889	14.0677	29.7450	47.3717	63.1329	72.4085
6.40	4.05874	15.9260	33.4851	52.8136	68.7803	78.3811
6.60	4.59342	17.9646	37.5256	58.2627	75.1024	85.2949
6.80	5.18679	20.1901	41.7718	64.1334	81.9818	90.4624
7.00	5.84259	22.7564	46.3579	70.4764	88.6351	94.0314
8.00			73.8045	105.415	121.132	
9.00			108.872	144.589	149.522	
10.00			150.845	184.262	169.786	
11.00			198.315	220.264	160.781	
12.00			249.113	248.926	184.585	
13.00			300.489	258.086	185.003	
14.00			349.537	277.684	186.[illegible]33	
15.00			385.636	280.031	195.137	